# METHODS OF SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER IN THE REAL DOMAIN

by

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# METHODS OF SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER IN THE REAL DOMAIN

Graduate Lecture Notes

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## ABSTRACT

A summary of the general theory of nonlinear differential equations of first order is given, with the aim of providing practical working rules for the analysis of technical problems, without pretense to rigor and completeness. In general, only equations of first order are considered here. After a discussion of the existing conditions and the analysis of singular points and of the integral solutions of the few types of equations which can be integrated in closed form, principal analytical and graphical procedures for the approximation of the solutions are described.

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## INTRODUCTION

The analysis of the d-c response of single energy nonlinear circuits was presented in a previous report.* It was shown that rigorous solutions in closed form could be obtained, and it was outlined that in this case several concepts and characteristics derived from linear analysis do not apply.

It is now proposed to extend the analysis to the a-c response of the same circuits. This, however, requires the application of special techniques of nonlinear analysis and presents difficulties of much higher order than the ones previously met. In general, no rigorous solutions in closed form can be obtained and one has to make recourse to approximation methods. For this reason, before proceeding to the analysis of the a-c response, which will be presented in a future report, it is convenient to summarise briefly the principal methods of analysis of linear and nonlinear differential equations which describe the behavior of single energy systems. The following notes, which are mainly limited to differential equations of first order, are only intended to give working rules to the student of nonlinear problems. The reader interested in rigorous and complete discussion of the subject is referred to the various excellent texts on differential equations published.**

## I. General Characteristics of Differential Equations of First Order: Existence of Solutions, Regular and Singular Points

A group of functions represented analytically by an expression of type

$$F(x,y,c) = 0 \tag{1}$$

where c is a variable parameter, is said to constitute a "family". The characteristic relationship of the functions of the family is an equation obtained by elimination of the parameter c between (1) and its derivative

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \tag{2}$$

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* Report R-271-52, PIB-210. This is the second of a series of reports on the analysis of nonlinear circuits, based on the material of the Graduate Course "Nonlinear Analysis" offered by Dr. E. Weber at the Polytechnic Institute of Brooklyn.

** See for instance: E. Picard, Traite' d'analyse, Gauthier-Villars (1905); H. Poincare', Les méthodes nouvelles de la méchanique céleste, E. Flammarion, Paris (1908); E. Goursat, Differential equations (Eng. Trans.) Boston (1917); E. L. Ince, Ordinary differential equations, Dover Publ. New York (1927); L. Bieberbach, Differentialgleichungen, Dover Publ. (1930); G. Sansone, Equazioni differenziali nel campo reale, Zanichelli, Italy (1940).

Such relationship of type

$$\Phi\,(x,y,p) = 0 \tag{3}$$

where $p = dy/dx$, is called a differential equation of first order. Similarly, a family of functions with two variable independent parameters is represented by a differential equation of second order. Therefore, in general, the order of the differential equation is equal to the number of arbitrary parameters upon which the family of functions depend, provided that these parameters are not mutually interdependent.

The reverse process of finding the various functions of the family from their differential equation is usually very difficult. In addition, it is clear that the differential equation may possess other integrals besides those represented by the family from which it has been derived.

For example, the circles in a plane form a three-dimensional family of equation

$$x^2 + y^2 + 2Ax + 2By + C = 0 \tag{4}$$

Differentiating three times and eliminating A, B, and C, one has

$$y'''(1+y'^2) - 3\,y'y''^2 = 0$$

This differential equation is satisfied not only by Eq. (4), but also by the equation of any straight line in the plane, since

$$y'' = y''' = 0$$

is a solution.

In the following we shall investigate the problem of the existence of solutions in a certain domain D(x,y) (such a domain is four-dimensional if x and y are considered complex variables), the characteristics of various types of singular points, and the methods of solution of the differential Eq. (3) with closed form or with approximate or numerical expressions.

## Regular and Singular Points

Given a differential equation of type (3) and a domain D(x,y) one is confronted with the problems of investigating a) the existence of solutions of (3) at each point of D, b) the nature of such solutions, i.e. whether or not they all belong to a family of type (1), and c) the characteristics of singular points of D(x,y).

Let us assume that Eq. (3) can be solved for $p = dy/dx$ and that, for any initial pair $(x_0,y_0)$ in D, there exists one and only one root

$$p = f(x,y) \qquad (5)$$

which reduces to $p_0$ when $x = x_0, y = y_0$. For greater generality we shall consider x and y as complex variables. Cauchy proved the following existence theorem. If, within a circle $|x-x_0| < h$, $f(x,y)$ is analytic in x and in y, then Eq. (5) possesses a unique solution $y = y(x)$ which is analytic within the circle and reduces to $y_0$ when $x = x_0$. In the case of real variables, Lipschitz simplified the proof of Cauchy, showing that the existing conditions reduce to a) the continuity of $f(x,y)$ within the rectangular domain

$$|x - x_0| \leq h \qquad |y - y_0| \leq b$$

where $h \leq b/M$ and M is the upper bound of $|f(x,y)|$ in the domain, and b) the existence of a positive number k, such that

$$|f(x,y') - f(x,y)| < k|y' - y|$$

where y and $y'$ are any two numbers of the rectangular domain.

The previous existence theorem is not applicable if Eq. (3) possesses a multiple p root for $x = x_0$, $y = y_0$. In this case, Eq. (3) is equivalent to m differential equations (where m is the order of multiplicity in p) and, in general, possesses m integrals at $x_0y_0$ called singular integrals. Such a situation arises at points on the envelope of the family of integral curves (1), at multiple points of any integral curve (1) where two branches of the same curve touch, and at tac points where two nonconsecutive curves (1) touch. The totality of loci for which at least two values of p are equal is obtained eliminating p between (3) and the equation $d\Phi/dp = 0$. If $R(x,y) = 0$ is the equation so obtained, this, in general, will not satisfy the given differential equation and, for this reason, will not be one of its integrals. In such case, $R(x,y) = 0$ is the locus of the multiple points or the locus of the tac points of the actual integral curves. If $R(x,y) = 0$ satisfies Eq. (3), then it represents the envelope of the family of integral curves.

To illustrate graphically this result, one can consider the family of isoclinals obtained from Eq. (3) by letting p = const. = k. A general survey of the integral solutions of Eq. (3) is obtained by sketching such a family as a function of the parameter k. If the coordinate plane (x,y) is

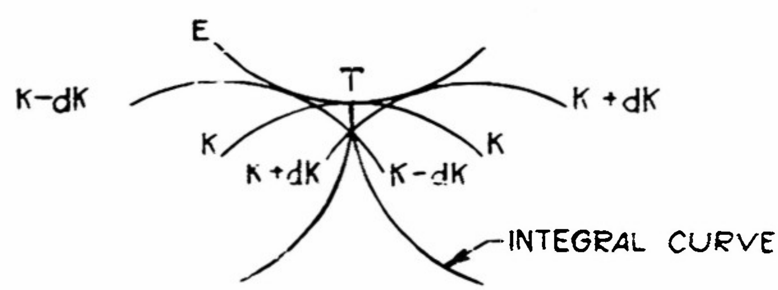


Fig. 1 - Envelope (E) of the family of isoclinal curves and locus of cusps.

covered with enough curves, it is possible to sketch integral curves starting at any initial point $(x_0,y_0)$ and proceeding in steps from $(x,y)$ to $x+dx, y+dy$ in the direction of the isoclinal passing through $(x,y)$. In Fig. 1 is represented the envelope E of isoclinal curves and indicated that it is the locus of cusps of the integral curves. As a matter of fact, in general, the isoclinal curves have a slope different than the values of k for which they are defined. There follows that, in general, integral curves cross the isoclinal curves. However, at the envelope E, which also has a slope different than p, the integral curves cannot cross since there are no contiguous isoclinal curves on the other side of the envelope. Consequently, the integral curves have a cusp or a stop point on the envelope.

The equation $R(x,y) = 0$ can also include a locus of double points of the isoclinal family. The case is indicated in Fig. 2 where it appears that the corresponding curve is the locus of tac points of the integral curves. As

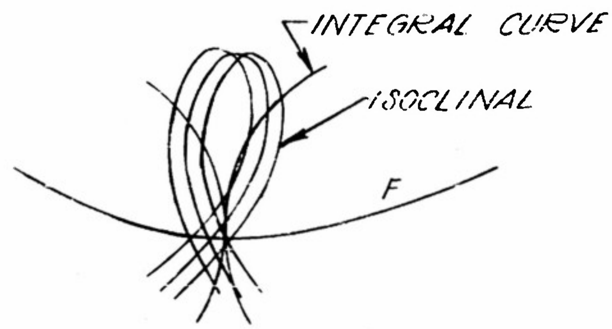


Fig. 2 - Tac locus (F) of integral curves, and locus of nodes of isoclinal curves.

a matter of fact, for each point of the curve F there are two possible directions of the corresponding integral curves, and therefore F is the locus of tangency where 2 nonconsecutive curves touch.

In order to find the locus of multiple points of the isoclinals, it is necessary and sufficient that the slope of the isoclinals, defined by

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = 0$$

is satisfied by two different values of dy/dx if

$$\frac{\partial \Phi}{\partial x} = 0, \quad \frac{\partial \Phi}{\partial y} = 0.$$

Finally, let us consider $(x,y)$ pairs for which Eq. (3) does possess a unique root p, but the existence conditions are not satisfied. Such pairs are called singular points of the equation and, in general, are isolated.

They play an important part in characterizing the behavior of the various integral curves in their vicinity because they are themselves singular points of the solutions, i.e., points at which the solutions are discontinuous, not unique, or not existent.

Let us assume that in Eq. (5), $f(x,y)$ is a rational function of $y$, i.e.,

$$\frac{dy}{dx} = f(x,y) = \frac{P(x,y)}{Q(x,y)} \tag{6}$$

where $P(x,y)$, $Q(x,y)$ are polynomials in $x,y$ with no common polynomial divisor

$$P(x,y) = P_o(x) + yP_1(x) + y^2P_2(x) + \dots + y^mP_m(x)$$

$$Q(x,y) = Q_o(x) + yQ_1(x) + y^2Q_2(x) + \dots + y^nQ_n(x)$$

The singularities of (6) are discrete and may be separated into two fundamental classes, the fixed or intrinsic and the movable or parametric singularities. The first ones arise at points $x$, where a) any of the coefficients $P_i$ or $Q_i$ has a singularity which cannot be removed by multiplying $P(x,y)$ and $Q(x,y)$ times an appropriate function of $x$; b) $Q(x,y)$ is identically zero; and c) $Q(x_1,y) = 0$, $P(x_1,y) = 0$ are satisfied simultaneously by a particular value of $y$.

The intrinsic singularities are connected with only some of the functions $F(x,y,c) = 0$ and arise in correspondence of values $x_1$ where such functions have a multiple point. For this reason, they depend upon the value of $c$, i.e., upon the initial conditions. Parametric singularities are found only in nonlinear differential equations; they can coexist with intrinsic singularities.

For example, the solution of the differential equation

$$y' = y^2$$

with $y(0) = y_o$ is

$$y = y_o/(1 - xy_o)$$

This integral has a pole at $x = 1/y_o$.

It may be shown that an equation of the first order and second degree of the Riccati type

$$\frac{dy}{dx} = P_o(x) + yP_1(x) + y^2P_2(x) \tag{7}$$

cannot possess multiple point singularities. There follows that linear equations of first order and nonlinear Riccati equations possess solutions which are rational functions of the initial value $y_o$, i.e.,

$$y = \frac{y_o A(x) + B(x)}{y_o C(x) + D(x)}$$

A Riccati equation can be converted into a linear homogeneous equation of second order with the substitution

$$y = -\frac{u'}{u P_2(x)}$$

One obtains from (7)

$$P_2(x) \frac{d^2u}{dx^2} - \left[ P'(x) + P_1(x) P_2(x) \right] \frac{du}{dx} + P_o(x) P_2^2(x)u = 0$$

and the solution of this differential equation is, in general, of type

$$u = C_1 u_1(x) + C_2 u_2(x).$$

It may be shown* that the movable singular points of an equation of type (6) can only be poles or algebraic critical points.

## Classification of Intrinsic Singular Points

We shall now examine the methods for the determination of the characteristics of the intrinsic singular points of Eq. (6), and, in particular, those which are common zeros of $P(x,y)$ and $Q(x,y)$, but are not stationary points of either function. If $(x_1,y_1)$ is one such pair, expanding $P(x,y)$ and $Q(x,y)$ in its vicinity by Taylor series

$$\varphi(x,y) = \varphi(x_1\, y_1) + \left[ (x-x_1)\varphi'_x(x_1y_1) + (y-y_1)\varphi'_y(x_1y_1) \right] +$$

$$+ \frac{1}{2!} \left[ (x-x_1)^2 \varphi''_{xx}(x_1y_1) + 2(x-x_1)(y-y_1)\varphi''_{xy}(x_1y_1) + (y-y_1)^2 \varphi''_{yy}(x_1y_1) \right]$$

$$+ \ldots.$$

one can write Eq. (5) as follows:

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* Painlevé - Équations différentielles ordinaires. Encycl. des Sciences math. t. 2, V.3, 1910.

$$\frac{dy}{dx} = \frac{a(x-x_1) + b(y-y_1) + P_1(x,y)}{c(x-x_1) + d(y-y_1) + Q_1(x,y)} \tag{8}$$

In Eq. (8), $a = \left(\frac{\partial P}{\partial x}\right)_{x_1y_1}$, $b = \left(\frac{\partial P}{\partial y}\right)_{x_1y_1}$, $c = \left(\frac{\partial Q}{\partial x}\right)_{x_1y_1}$, $d = \left(\frac{\partial Q}{\partial y}\right)_{x_1y_1}$, and $P_1(x,y)$, $Q_1(x,y)$ are polynomials in $(x-x_1)$, $(y-y_1)$ of degree not less than two. If the differences $(x-x_1)$,$(y-y_1)$ are made to approach zero along some arbitrary curve in $(x,y)$, $P_1(x,y)$ and $Q_1(x,y)$ will vanish to an order higher than the first so that, in a region sufficiently small surrounding $(x_1y_1)$, Eq. (8) may be written approximately

$$\frac{dy}{dx} \simeq \frac{a(x-x_1) + b(y-y_1)}{c(x-x_1) + d(y-y_1)} \tag{9}$$

Eq. (9) is homogeneous in $(x-x_1)$, $(y-y_1)$ and, by means of the substitution $y-y_1 = z(x-x_1)$, may be reduced to one whose variables are separable. Neglecting the case $ad - bc = 0$ for which equation (9) becomes $p$ = const., it is found that the characteristic of the singularity $(x_1y_1)$ depends upon the nature of the roots $z_1$, $z_2$ of the algebraic equation

$$z^2 - (b+c)\, z - (ad-bc) = 0$$

i.e., upon the discriminant $\Delta = (b-c)^2 + 4\, ad$. If $\Delta < 0$, $(x_1y_1)$ is a limit point (focus) of the integral curves which are spiral-like; if $\Delta \geq 0$ but $ad - bc < 0$, $(x_1y_1)$ is an actual common point of the integral curves (node); if $\Delta > 0$ and $ad - bc > 0$, $(x_1y_1)$ is a saddle point.

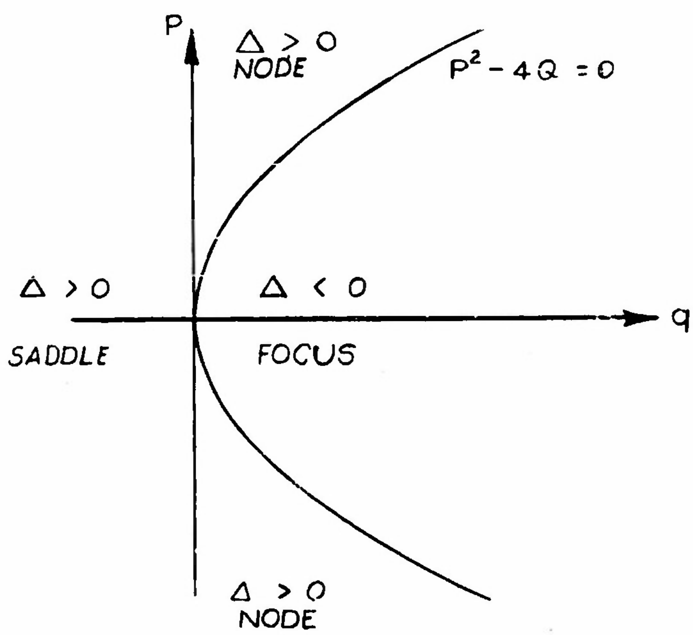


Fig. 3

Letting

$$P = -(b+c) = -(z_1+z_2), \quad Q = -(ad-bc) = z_1 z_2$$

one has

$$\Delta = P^2-4Q$$

so that the previous conditions in a P, Q plane correspond to the regions outlined in Fig. 3.

It is possible to assign an index to the various types of singularities. As a matter of fact, if in the (x,y) plane we consider an arbitrary closed curve C which does not possess any multiple points and surrounds one and only one singularity, the total number of revolutions N made by the vector of components P(x,y), Q(x,y) in a complete circuitation of C is +1 if the singularity is a node or a focus, -1 if it is a saddle point. The number N is called index of the singularity; if the curve C encloses several singularities, the value of N is the sum of their indexes. This number is expressed mathematically with the relation

$$N = \frac{1}{2\pi}\oint_c d \tan^{-1}\frac{Q(x,y)}{P(x,y)} = \frac{1}{2\pi}\oint_c \frac{PdQ - QdP}{P^2 + Q^2}$$

As a consequence of the previous statements there follows that if the curve C is an actual solution of Eq. (9), the corresponding value of N is +1, so that the sum of the indexes of the singularities enclosed must add up to +1.

The subject of singularities of a differential equation of first order will be discussed further in a subsequent report, to consider cases in which x and y are both functions of a third variable, the time. However, we shall indicate here briefly the case of singularities of higher order. These correspond to points (x', y') which are not only common zeros of P(x,y) and Q(x,y), but also stationary points for them. For example, the equation

$$p = \frac{3x^2}{2y}$$

admits the general integral

$$y^2 = x^3 + C$$

This consists of a system of cubics of which the curve $y^2 = x^3$ has a cusp at the origin.

The equation

$$p = \frac{8xy^3}{-3x^4 + 6x^2y^2 + y^4}$$

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* [illegible] Dulac - Points singuliers des [illegible] mathém. Fasc. LXI - 193[illegible]. [illegible]

[illegible] Sansone - Equazioni differenziali [illegible] Bologna, Italy.

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For an equation of type (6), necessary and sufficient condition of integrability is that

$$P(x,y)\ dx - Q(x,y)\ dy = 0 \tag{11}$$

be an exact differential, i.e.,

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0$$

The solution is then expressed with

$$\int_{x_o}^{x} P\ dx + \int_{y_o}^{y} Q\ dy = c$$

The condition of integration is always verified when the variables are separable, i.e., $P = P(x)$, $Q = Q(y)$, or vice versa.

If $P(x,y)$, $Q(x,y)$ are homogeneous functions of x and y of the same degree, the separation of variables can be achieved with the substitution $y = zx$ which transforms (10) into

$$\left[P(1,z) + z\ Q(1,z)\right] dx - x\ Q(1,z)\ dz = 0$$

More generally, if (11) is not an exact differential, it may be transformed into one by multiplication with an integrating factor $\mu(x,y)$, such that

$$\mu(P\ dx - Q\ dy) = 0$$

is a total differential. There exists an infinity of integrating factors which are solutions of the partial differential equation

$$\frac{\partial(\mu P)}{\partial y} + \frac{\partial(\mu Q)}{\partial x} = 0$$

i.e.,

$$P\frac{\partial\mu}{\partial y} + Q\frac{\partial\mu}{\partial x} + \mu\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) = 0 \tag{12}$$

The solution of (12) is in itself a problem more difficult than the solution of (11), but fortunately we only need to find a particular integral of (11) and this, in certain cases, can be found easily. As an example, consider any linear differential equation of first order

$$\frac{dy}{dx} + a_1(x)\ y = b_1(x) \tag{13}$$

In this case we have $P = b_1 - a_1 y$, $Q = 1$, and Eq. (12) becomes

$$(b_1 - a_1 y)\frac{\partial\mu}{\partial y} + \frac{\partial\mu}{\partial x} = \mu\, a_1$$

A particular integral of this equation is $\mu = \mu(x)$ obtained from $\frac{d\mu}{dx} = \mu a$, i.e.,

$$\mu(x) = \varepsilon^{\int a_1(x)dx}$$

With the use of this integrating factor, Eq. (13) is transformed into the exact differential

$$\varepsilon^{\int a_1 dx}(b_1 - a_1 y)\,dx - \varepsilon^{\int a_1 dx}\,dy = 0$$

whose general integral according to (12) is

$$y - y_o = \varepsilon^{-\int a_1 dx}\left\{C + \int_{x_o}^{x}(-a_1 y + b)\,\varepsilon^{\int a_1 dx}\,dx\right\}$$

i.e.,

$$y = C\,\varepsilon^{-\int a_1 dx} + \varepsilon^{-\int a_1 dx}\int_{x_o}^{x} b_1(x)\,\varepsilon^{\int a_1 dx}\,dx$$

In the case of the nonlinear Bernoulli equation

$$\frac{dy}{dx} + a_1(x)\,y + a_n(x)\,y^n = 0 \tag{14}$$

Eq. (12) becomes

$$(a_1 y + a_n y^n)\frac{\partial\mu}{\partial x} + \frac{\partial\mu}{\partial x} + \mu(a_1 + na_n\,y^{n-1}) = 0$$

A particular solution of this equation is not easily found. However, Eq. (14) can be transformed into a linear equation by means of the substitution $z = y^{1-n}$ and becomes

$$\frac{dz}{dx} + (1-n)(a_1 z + a_n) = 0$$

The Riccati equation

$$\frac{dy}{dx} + a_1\,y + a_2\,y^2 = b_1 \tag{15}$$

which contains the Bernoulli equation as a particular case, cannot be integrated by a general method. Its solution is simplified, however, if any

$$(b_1 - a_1 y)\frac{\partial\mu}{\partial y} + \frac{\partial\mu}{\partial x} = \mu a_1$$

A particular integral of this equation is $\mu = \mu(x)$ obtained from $\frac{d\mu}{dx} = \mu a$, i.e.,

$$\mu(x) = \varepsilon^{\int a_1(x)dx}$$

With the use of this integrating factor, Eq. (13) is transformed into the exact differential

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whose general integral according to (12) is

$$y - y_o = \varepsilon^{-\int a_1 dx}\left\{C + \int_{x_o}^{x} (-a_1 y + b)\, \varepsilon^{\int a_1 dx}\, dx\right\}$$

i.e.,

$$y = C\, \varepsilon^{-\int a_1 dx} + \varepsilon^{-\int a_1 dx} \int_{x_o}^{x} b_1(x)\, \varepsilon^{\int a_1 dx}\, dx$$

In the case of the nonlinear Bernoulli equation

$$\frac{dy}{dx} + a_1(x)\, y + a_n(x)\, y^n = 0 \tag{14}$$

Eq. (12) becomes

$$(a_1 y + a_n y^n)\frac{\partial\mu}{\partial x} + \frac{\partial\mu}{\partial x} + \mu(a_1 + n a_n y^{n-1}) = 0$$

A particular solution of this equation is not easily found. However, Eq. (14) can be transformed into a linear equation by means of the substitution $z = y^{1-n}$ and becomes

$$\frac{dz}{dx} + (1-n)(a_1 z + a_n) = 0$$

The Riccati equation

$$\frac{dy}{dx} + a_1 y + a_2 y^2 = b_1 \tag{15}$$

which contains the Bernoulli equation as a particular case, cannot be integrated by a general method. Its solution is simplified, however, if any particular integral $y^*$ is known. As a matter of fact, letting $y = y^* + z$ the equation is changed into the Bernoulli type

$$\frac{dz}{dx} + (2y^* a_2 + a_1)\, z + a_2 z^2 = 0$$

Passing now to other possible forms of Eq. (3), a solution can be obtained at least theoretically when $\Phi(x,y,p)$ is a polynomial of degree n in p. As a matter of fact, writing the equation in factorized form

$$\Phi(x,y,p) = (p-\alpha_1)(p-\alpha_2) \;\ldots\; (p-\alpha_n) = 0, \qquad (16)$$

one can show that the general integral is the product of the integrals of the equations obtained equating to zero each factor of (16).

Other cases of interest are those in which $\Phi(x,y,p)$ can be written in one of the forms

$$x = f_1(p) \qquad (17)$$

$$y = f_2(p) \qquad (18)$$

$$y = xf_3(p) \qquad (19)$$

The solutions of these equations can be written in closed parametric form in terms of p. One has respectively

$$x = f_1(p), \quad y = c + \int pf'(p)\, dp \qquad (17')$$

$$y = f_2(p), \quad x = c + \int \frac{1}{p} f_2'(p)\, dp \qquad (18')$$

$$y = xf_3(p), \quad x = c \exp \left\{ \int \frac{f_3'(p)\, dp}{p - f_3(p)} \right\} \qquad (19')$$

Similarly, when $\Phi(x,y,p)$ is linear in x and in y, of type

$$\Phi(x,y,p) = x\,\varphi(p) - y + \psi(p)$$

with $\varphi(p) \neq p$, the solution is obtained in parametric form as

$$x = f_4(p), \quad y = f_5(p)$$

where $f_4(p)$ is the integral of the linear equation

$$\frac{dx}{dp} - \frac{\varphi'(p)}{p-\varphi(p)}\, x = \frac{\psi'(p)}{p-\varphi(p)}$$

and $f_5(p) = x\varphi(p) + \psi(p)$.

On the other hand, the roots $p_i$ of

$$p - \varphi(p) = 0$$

satisfy the condition $dp/dx = 0$ and furnish a number of singular solutions of type

$$y = x\varphi(p_i) + \psi(p_i)$$

In certain cases a given differential equation can be transformed into an integrable form with a substitution of variable. For instance, the transformation of Legendre

$$x = P, \quad y = XP - Y, \quad p = X$$

transforms $\Phi(x,y,p) = 0$ into the equivalent equation

$$\Phi(P, XP-Y, X) = 0$$

## III. Methods for the Determination of Approximate Integral Solutions

Unfortunately, in the great generality of cases, a rigorous solution of a nonlinear differential equation cannot be found and for this reason it is necessary to have recourse to methods of approximation. These generally apply only to a differential equation of type (5)

$$p = f(x,y)$$

and in a domain D in which $f(x,y)$ is analytic in x and y. They provide the solution as a limit of a sequence of functions, or as sum of an infinite series, or as a numerical expression.

### a) Method of Successive Approximations (Iteration)

If $x_o$, $y_o$ is a point of the domain D, the solution of (5) satisfies the integral equation

$$y = y_o + \int_{x_o}^{x} f(\xi, y(\xi))\, d\xi \qquad (20)$$

where $y(\xi)$ is unknown and x is chosen within D. Let us consider the sequence

$$y_1 = y_o + \int_{x_o}^{x} f(\xi, y_o)\, d\xi$$

$$y_2 = y_o + \int_{x_o}^{x} f(\xi, y_1)\, d\xi$$

. . . .

. . . .

It can be shown that this sequence possesses a limit for $n \to \infty$, and that this limit is the desired solution of (5). While this procedure is very general, the convergence of the sequence obtained is rather slow. In order to improve the latter, in special cases modifications can be introduced in the method. Suppose, for instance, that Eq. (5) is of the form

$$p = f_1(x,y) + f_2(x,y) \tag{21}$$

where $\varepsilon f_2 << f_1$ within the domain D. If a solution of

$$p = f_1(x,y)$$

can be found, one assumes it as zero approximation and obtains, the first approximation solving

$$p = f_1(x,y) + \varepsilon f_2(x,y_o)$$

Similarly, the second approximation is obtained as solution of

$$p = f_1(x,y) + \varepsilon f_2(x,y_1)$$

and so on.

Another modification of the method is obtained considering the equation

$$\frac{dx}{dy} = \frac{1}{f(x,y)} \tag{22}$$

which sometimes may be integrated more readily. One has similarly

$$x = x_o + \int_{y_o}^{y} \frac{d\xi}{f(x(\xi), \xi)}$$

and correspondingly the approximation sequence is

$$x_1 = x_o + \int_{y_o}^{y} \frac{d\xi}{f(x_o, \xi)}$$

$$x_2 = x_0 + \int_{y_0}^{y} \frac{d\xi}{f(x_1, \xi)}$$

. . . . . . .

b) Method of Integration in Series (Perturbation)

While with the previous method the solution is expressed as limit of a sequence, it can also be expressed as sum of a converging infinite series. In a domain of analyticity of $f(x,y)$ surrounding the initial pair $(x_0,y_0)$ one can compute all successive derivatives of y at $(x_0,y_0)$ by means of the given differential equation. Therefore, by Taylor series, one has

$$y = y_0 + y_0'(x-x_0) + y_0''\frac{(x-x_0)^2}{2!} + \dots \qquad (23)$$

In practice the computation of successive derivatives may be rather cumbersome. If one accepts the principle that y can be expressed as a power series, its coefficients can be obtained by substituting such an expression with indeterminate coefficients for the dependent variable in the differential equation. Since it must make the equation an identity, one can readily obtain recurrence relations between the coefficients of the power series by comparing terms involving the same power of the variable.

The method of solution in series may be applied also in the neighborhood of some types of singular points of the equation, namely, those in correspondence of which the solution is expressed in the form

$$y = (x-x_0)^r\left[a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots\right] \qquad (24)$$

with $a_0 \neq 0$ and r any real number. To investigate the possible existence of such type of solution, one substitutes the series (24) into the differential equation and equates to zero the coefficient of the term of lowest degree in $(x-x_0)$. If this coefficient is independent of r, the expression (24) cannot be used to represent the solution in the neighborhood of $(x_0,y_0)$. Otherwise, one determines r from this equation and then proceeds to the evaluation of all other coefficients.

When the differential Eq. (5) can be written in the form

$$p = f(x,y) = f_1(x,y) + \varepsilon f_2(x,y)$$

with $\varepsilon f_2 \ll f_1$, it is possible to express the solution with an infinite series whose terms are not simple powers of the independent variable, but more general functions of such variable, simultaneously defined in a certain interval of x. For this purpose, it is observed that in the given domain $f(x,y)$ can be considered an analytic function not only of the (complex) variables x and y, but also of $\varepsilon$; therefore, the solution can be expressed in a series of powers of $\varepsilon$, i.e.,

$$y = y_o(x) + \varepsilon\, y_1(x) + \varepsilon^2\, y_2(x) + \dots \qquad (25)$$

Substituting the expression (25) into the differential equation, one obtains

$$\dot{y}_o + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2 + \dots = f_1(x, y_o + \varepsilon y_1 + \dots) + \varepsilon f_2(x, y_o + \varepsilon y_1 + \dots)$$

Now, equating the terms of same degree in $\varepsilon$, one obtains a system of recurrence equations which permits the determination of the functions $y_o$, $y_1$, $y_2$ .....
For instance, $y_o$ is the solution of the equation

$$\dot{y}_o = f_1(x,y)$$

This special case of the method of integration by series is known as "perturbation method". A close analogy exists between perturbation and iteration methods; depending upon the particular differential equation, one or the other may be more convenient, but in general both become rather cumbersome after the second or third approximation.

In the application of each of the methods previously discussed, it is necessary to satisfy the given initial or boundary condition. Since the equations here considered are of the first order, this condition reduces to one relation, for instance, the initial value $y(0)$. In addition, if it is known that the equation possesses a periodic solution (in response to an external driving term, or as free solution of a degenerate system), a periodic condition of the type $y(x + 2\pi) = y(x)$ must be also satisfied. A degenerate system is one which is represented by a differential equation of second order, which, on account of the relative smallness of the coefficient of the second derivative or of that of the function y (spring constant) reduces actually to a differential equation of first order. A system of the type

$$\frac{dy}{dx} - f(y) = 0 \qquad (26)$$

possesses a periodic solution if the corresponding path in the plane (dy/dx, y)

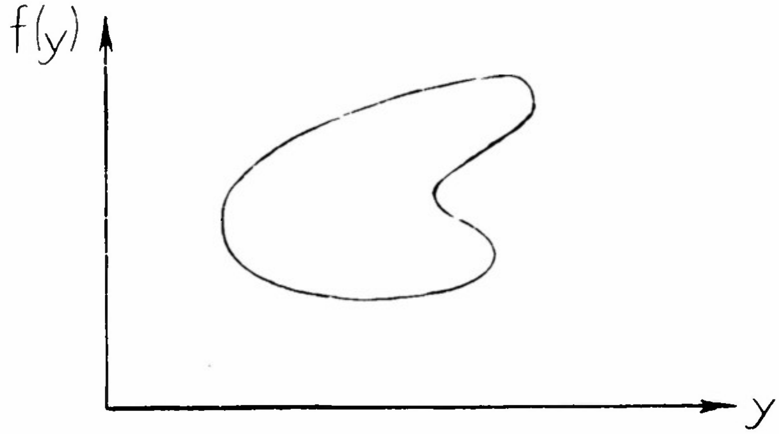


Fig. 4 - Multivalued Function

is a closed curve. This implies that $f(y)$ must be a multivalued function (Fig. 4) since, for some values of $y$, $f(y)$ will assume at least two different values on the solution path. Thus, the function $f(y)$ in general will be non-analytical, including radicals or discontinuities.*

By means of the methods of approximation described, the solution is expressed as an infinite series. In order to satisfy the initial condition it is sufficient to impose that the first term of such a series assumes the required initial value and all other terms assume correspondingly zero initial value. In addition, in order to satisfy the periodicity condition, when this is required, one must equate to zero all the coefficients of terms in the series, possessing a periodicity other than the required one.

c) Method of Sinusoidal Analysis

When it is known that the solution presents periodic terms of a certain frequency, one can express $y$ as a Fourier series in $x$ and determine its coefficients by direct substitution into the differential equation. By this procedure, the integral of the nonlinear differential equation is obtained from the solution of a system of equations which are not of differential type, but are of algebraic or transcendental type depending upon the nature of the nonlinearity existing in the original differential equation. For instance, given

$$\frac{dy}{dt} - hy = f(y,t) \tag{27}$$

where it is known that $y$ is a periodic function of $t$, one can assume in general

$$y = \sum_{n=1}^{\infty} (a_n \sin n\omega t + b_n \cos n\omega t)$$

$$y' = \sum_{n=1}^{\infty} (\omega n\, a_n \cos n\omega t - \omega n\, b_n \sin n\omega t)$$

There follows that

$$f(y,t) = \sum_{1}^{\infty} (f_{1n} \sin n\omega t + f_{2n} \cos n\omega t)$$

where

$$f_{1n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y,t) \sin n\omega t \, d\omega t$$

$$f_{2n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y,t) \cos n\omega t \, d\omega t$$

* Andronov, Chaikin - Theory of Oscillations, Princeton Un. Press 1949, p. 138, Chapter IV.

Clearly the solution of the equations obtained by identifying coefficients of isofrequential terms is impossible in the general case. If, however, it is permitted to assume that only a limited number of harmonics exists, then the equations may be simplified conveniently to permit the solution for the values of the various coefficients of the Fourier series.

In particular one might combine the foregoing procedure with a procedure of perturbation*. For instance, one assumes that as first approximation y can be expressed by means of its fundamental term only, i.e., $y = a_1 \sin \omega t + b_1 \cos \omega t$. Consequently, Eq. (27) reduces to

$$\frac{dy}{dt} - hy = f_{11} \sin \omega t + f_{21} \cos \omega t$$

where

$$f_{11} = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left[a_1 \sin \omega t + b_1 \cos \omega t,\ t\right] \sin \omega t \, d\omega t$$

$$f_{21} = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left[a_1 \sin \omega t + b_1 \cos \omega t,\ t\right] \cos \omega t \, d\omega t$$

Letting now $y = \mathrm{Re}\,\bar{Y}$, where $\bar{Y} = (b_1 + ja_1)\,\varepsilon^{j\omega t} = \bar{A}\,\varepsilon^{j\omega t} = A\,\varepsilon^{j\varphi}\,\varepsilon^{j\omega t}$, one can write equivalently

$$\left[j\omega - h\right] \bar{Y} = (f_{21} + jf_{11})\,\varepsilon^{j\omega t} = \frac{f_{21} + jf_{11}}{A\,\varepsilon^{j\varphi}}\,\bar{Y}$$

i.e.,

$$\left[j\omega - h\right] \bar{Y} = N\left[\bar{A}\right] \bar{Y} \tag{28}$$

If the nonlinear term f(y,t) contains also derivatives, the resultant expression N will be a function of $\bar{A}$ and $j\omega$, i.e., $N(\bar{A}, j\omega)$.

The term $N(\bar{A}, j\omega)$ is called describing function of the nonlinear expression f(y,t); it is seen that it is a function of the complex amplitude $\bar{A} = Ae^{j\varphi}$ and of the frequency $\omega$. If f(y,t) is linear in y, the corresponding describing function becomes the transfer function. In general, we have

$$f(y,t) - \mathrm{Re}\,N(\bar{A}, j\omega)\,\bar{Y} = \mu\Phi(y,t) \tag{29}$$

or identically

$$f(y,t) - \left[N(\bar{A}, j\omega)\,\bar{Y} + N(\bar{A}^*, j\omega)\,\bar{Y}^*\right] = \mu\Phi(y,t)$$

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* B. V. Bulgakov - Periodic processes in free pseudo-linear oscillatory systems, Jour. Franklin Inst., Vol. 235, June 1943, p. 591-616.

E. C. Johnson - Sinusoidal analysis of feedback control systems containing nonlinear elements. Trans. AIEE, July 1952.

The present procedure of approximate solution is valid when in (29) one can consider $\mu\Phi(y,t)$ very small and neglect it in first approximation. Then the amplitude $Ae^{j\varphi}$ is determined graphically or analytically solving

$$j\omega - h = N(\bar{A}, j\omega).$$

A better approximation to the actual solution can be obtained when the difference (29) is small with respect to the other terms in the differential equation, by writing

$$\frac{dy}{dt} - hy - \text{Re}\, N(\bar{A}, j\omega)\, \bar{Y} = \mu\Phi(y,t) \qquad (30)$$

In this relation $\mu$ is a coefficient indicative of smallness. Now following a perturbation method we imagine to expand the true solution y into a power series in $\mu$, i.e.,

$$y = y_o + \mu y_1 + \frac{\mu^2}{2!} y_2 + \ldots$$

The first approximation is obtained as indicated beforehand letting $\mu = 0$. To obtain the second approximation, one differentiates Eq. (30) with respect to $\mu$ and then lets $\mu = 0$. There follows

$$\frac{dy_1}{dt} - hy_1 - \text{Re}\, N(\bar{A}, j\omega)\, \bar{Y}_1 = \Phi(y_o) \qquad (31)$$

Expanding $\Phi(y_o)$ in Fourier series one has

$$\Phi(y_o) = \frac{1}{2} \sum_{-\infty}^{+\infty} \Phi_n\, \varepsilon^{jn\omega t}$$

where

$$\Phi_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(y_o)\, \varepsilon^{-jn\omega t}\, d\omega t$$

There follows for the second member of (31)

$$\frac{dy_1}{dt} - hy_1 - \text{Re}\, N(\bar{A}, j\omega)\bar{Y}_1 = \frac{1}{2}\Phi_o + \frac{1}{2} \sum_{\substack{-\infty \\ n \neq 0, \pm 1}}^{+\infty} \Phi_n\, \varepsilon^{jn\omega t}$$

Now the condition of periodicity requires that $\Phi_1 = \Phi_{-1} = 0$. This condition is already verified, however, on account of (29). Therefore, the term of second approximation is obtained as

$$y_1 = \text{Re}\, \bar{A}_1 \varepsilon^{j\omega t} + \frac{1}{2} \sum_{n=2}^{\infty} \bar{Y}_{1n} \varepsilon^{jn\omega t} + \frac{1}{2} \sum_{n=2}^{-\infty} \bar{Y}_{2n} \varepsilon^{-jn\omega t}$$

It contains a new arbitrary constant $\bar{A}$ which must be determined on the basis of the initial condition, as indicated previously.

## IV. Linear Differential Equations with Variable Coefficients

The methods of solution described for nonlinear equations can be used also in the case of linear equations. However, in the latter case, since superposition applies, it is possible to make recourse to special methods of analysis, based on the use of the Green's function and of linear transforms. For greater generality we shall consider a linear differential equation of order n of type

$$\left[a_n(t)p^n + \dots + a_1(t)p + a_o(t)\right] y(t) = \left[b_m(t)p^m + \dots + b_o(t)\right] u(t) \quad (32)$$

which we can indicate for brevity with

$$L(p,t)\, y(t) = k(p,t)\, u(t) = r(t) \quad (33)$$

where $u(t)$, $r(t)$ are known functions (driving terms). In general, we also have a set of n linearly independent relations in $y(t)$ which represent the boundary conditions. The system

$$\begin{cases} L(p,t)\ y = r(t) \\ u_i(t) = \gamma_i \qquad\qquad i = 1.\ 2 \dots n \end{cases} \quad (34)$$

might not possess a solution not identically zero which together with its n-1 derivatives is continuous throughout the interval (a,b) of the independent variable. However, one can always find a function which formally satisfies the system (34) but violates at least in part the condition of continuity. In particular when $r(t) = 0$, $\gamma_i = 0$ (homogeneous system), such a solution is called Green's function $G(t, \tau)$. Such a function is continuous and possesses continuous derivatives of orders up to and including n-2 when $a \leq t \leq b$; in addition, it is such that its derivative of order n-1 is discontinuous at a point $\tau$ within (a,b), and presents there a jump upward $1/a_n(\tau)$; finally, it satisfies the given system at all points of (a,b) except at $t = \tau$.

If $G(t,\tau)$ is a solution of

$$\begin{cases} L(p,t)\ y = 0 \\ u_i(y) = 0 \qquad\qquad i = 1 \dots n \end{cases} \quad (35)$$

then an explicit solution of the nonhomogeneous system

$$\begin{cases} L(p,t)\ y = r(t) \\ u_i(y) = 0 \qquad\qquad i = 1 \dots n \end{cases} \quad (36)$$

is

$$y(t) = \int_a^b G(t,\tau)\, r(\tau)\, d\tau \tag{37}$$

The solution of the more general nonhomogeneous system (34) is

$$y(t) = \int_a^b G(t,\tau) r(\tau) d\tau + \gamma_1 G_1(t) + \ldots + \gamma_n G_n(t) \tag{38}$$

where $G_i(t)$ are the unique solutions of the system

$$\begin{aligned} L(G_i) &= 0 \\ U_j(G_i) &= 0 \\ U_i(G_i) &= 1 \qquad j = 1 \,..\, i-1, i+1 \,..n \end{aligned} \tag{39}$$

When the equations of the boundary conditions $U_i(y) = \gamma_i$ are expressed in diagonal form

$$y^{(v)}(o) = y_v \qquad v=0 \,..\, n-1$$

the latter procedure provides as functions $G_i(t)$ the solutions of

$$L(p,t)y = \left[a_n(o)y_{i-1} + \ldots + a_{n-i+1}(o)y_o\right] \delta^{n-i}(t) \tag{40}$$

Therefore, in this case, the final solution (38) may be considered as corresponding to a system

$$\begin{aligned} L(p,t)y &= r(t) + a_n(o)y_o \delta^{n-1}(t) + \left[a_n(o)y_1 + a_{n-1}(o)y_o\right] \delta^{n-2}(t) + \\ &+ \ldots + \left[a_n(o)y_{n-1} + \ldots + a_1(o)y_o\right] \delta(t) \\ U_i(y) &= 0 \qquad i = 1. \,..n \end{aligned} \tag{41}$$

In general, the rigorous solution of a variable linear system is not known. Approximate expressions for it can be obtained by application of methods of iteration or perturbation. For instance, writing the system (34) as

$$\begin{aligned} L_1(p,t)y &= L_2(p,t)y + r(t) \\ U_i(y) &= V_i(y) + \gamma_i \end{aligned} \tag{42}$$

where parts of a differential expression and of the boundary conditions have been transferred to the right hand side of the equations, one can determine uniquely a sequence of functions

$$y_o(t),\ y_1(t)\ \ldots\ y_n(t)\ \ldots$$

where $y_o(t)$ is such that $L_2(y_o)$ is continuous in (a,b) and $v_i(y_o)$ are finite, and $y_1(t),\ldots y_n(t)$ satisfy the recurrence relations

$$\begin{cases} L_1(y_r) = L_2(y_{r-1}) + r(t) \\ U_i(y_r) = V_i(y_{r-1}) + \gamma_i \qquad i = 1\ \ldots n \end{cases} \tag{43}$$

One should observe the relative arbitrariness of choice of $y_o(t)$. Another iteration method* consists in replacing the variable coefficients in L(p,t) with their mean values in the range (a,b) of the independent variable. Eq. (33) is then written

$$\bar{L}(p)y = k(p,t)u(t) + \Delta L(p,t)\, y$$

where

$$\bar{L}(p) = \bar{a}_n p^n + \ldots + \bar{a}_o$$

$$\Delta\bar{L}(p) = \left[\bar{a}_n - a_n(t)\right] p^n + \ldots + \left[\bar{a}_o - a_o(t)\right]$$

Starting with $y_o(t)$ equal to the solution of

$$\bar{L}(p).y(t) = k(p,t)\ .\ u(t)$$

one institutes recurrence relations of type (42) for the Eq. (43).

The investigations of variable linear differential equations can also be carried out in the complex frequency domains. However, direct transformation of (32) leads, in general, to another variable linear differential equation in which order in p and degree in s are interchanged with respect to Eq. (32). For this reason, in general, the new differential equation is not easier to integrate than the original one. However, one can find the Green's function of (32) or more generally the impulsive solution of (32) $W(t,\tau)$ when $u(t) = \delta(t)$ and transform it into the complex domain by means of

$$H(s,t) = \int_0^t W(t,\tau)\ \varepsilon^{-s(t-\tau)} d\tau$$

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* S. A. Schelkunoff, M. C. Gray - B.S.T.J., Vol. 27, April 1948, p. 350-364.

It may be shown* that the general solution of the systems (36) can be expressed by means of

$$y(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} H(s,t).U(s)\, \epsilon^{st} ds$$

where the product $H(s,t).U(s)$ should be considered as the frequency domain operation corresponding to the time domain integration

$$H(s,t) = \int_0^{\infty} W(t,\tau)\, U(\tau)\, d\tau$$

which is the analogous of (37).

Recapitulating, the procedure of solution of Eq. (32) in the time domain consists of finding the Green's function $G(t,\tau)$ [or more generally, the impulsive response $W(t,\tau)$] and then finding the total solution $y(t)$ by application of the superposition integral in one of the forms (37) or (38). In the frequency domain, on the contrary, one finds the corresponding transform $H(s,t)$ of the Green's function (or of the impulsive response) and then obtains the transform of the total solution $y(t)$ by multiplication of the transforms $H(s,t)$ and $U(s)$. In the latter process one uses standard Laplace transform tables, considering $t$ in $H(s,t)$ as a constant parameter. It is also possible to extend to the analysis of variable systems many concepts familiar to fixed systems**; for instance, for the case of variable networks one can extend the concepts of impedance, admittance, gain, and various theorems of linear fixed circuit analysis.

In order to find the function $H(s,t)$, which might be considered as the system function of variable systems, one does not have to find first the impulsive response, but can solve directly the given differential equation in the variable $H(s,t)$. For this purpose one applies familiar rules of linear differential operators, letting in (32) $u(t) = e^{st}$ and, correspondingly, $y(t) = H(s,t).e^{st}$. It is found

$$L(p,t)\ .\ H(s,t)\, \epsilon^{st} = k(p,t)\, \epsilon^{st}$$

and equivalently

$$L(p+s,\ t)\ .\ H(s,t) = k(s,t) \tag{44}$$

Equation (44), when written out, reads

$$\left[\left[\frac{1}{n!}\ \frac{\partial^n L(s,t)}{\partial s^n}\right] p^n + \ldots + \left[\frac{\partial L(s,t)}{\partial s}\right] p + L(s,t)\right] H(s,t) = k(s,t) \tag{45}$$

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* L. A. Zadeh - Proc. I.R.E., Vol. 38, 1950, p. 291.

** L. A. Zadeh - J.A.P., Vol. 21, Nov. 1950, p. 1171.

It should be observed that Eq. (45), in general, is not easier to integrate than Eq. (32) since one has to use the same methods of iteration or perturbation mentioned previously. For instance, in applying the iteration procedure (42) one might choose as starting function $y_0(t)$ the function $H_0(s,t)$ defined as

$$H_o(s,t) = \frac{K(s,t)}{L(s,t)}$$

which might be considered as first approximation to $H(s,t)$ when the coefficients $a_i(t)$ of $L(p,t)$ do not vary appreciably over the duration of the impulsive response. This method corresponds to replacing the differential equation (45) with

$$L(s,t) \cdot H(s,t) = K(s,t) - \sum_{j=1}^{n} \frac{1}{j!} \frac{\partial^j L(s,t)}{\partial s^j} p^j H(s,t)$$

where all derivative terms have been moved from the left to the right hand side. The recurrence Eqs. (43) are correspondingly written

$$L(s,t) \cdot H_r(s,t) = K(s,t) - \sum_{j=1}^{n} \frac{1}{j!} \frac{\partial^j L(s,t)}{\partial s^j} p^j H_{r-1}(s,t)$$

## V. Some Examples of Application of Methods of Approximate Integration

In order to show the technique of application of methods of approximate integration we shall work out as examples the solution of some differential equations. For the purpose of checking the order of approximation obtained, we choose first a linear differential equation which can be also solved rigorously.

Specifically, we consider the linear differential equation

$$\frac{dy}{dt} + \left[a + jbs(t)\right] y = 1 \qquad (46)$$

where $s(t) = \sin \omega t$, $0<b<a$. Its rigorous solution is known and asymptotically (steady state) is given by the particular integral

$$y = \varepsilon^{-at-jb\int s(t)dt} \int_{-\infty}^{t} \varepsilon^{at+jb\int s(t)dt}\, dt$$

Using the Fourier series expansion

$$\varepsilon^{-j\frac{b}{\omega}\cos \omega t} = \sum_{-\infty}^{+\infty} (-j)^n J_n\left(\frac{b}{\omega}\right) \varepsilon^{jn\omega t}$$

one has

$$\int_{-\infty}^{t} \varepsilon^{at-j\frac{b}{\omega}\cos\omega t}\,dt = \int_{-\infty}^{t} \varepsilon^{at} \sum_{-\infty}^{\infty} (-j)^n J_n(\tfrac{b}{\omega})\, \varepsilon^{jn\omega t}\,dt$$

Therefore, the asymptotic solution of (46) is written

$$y(t) = \varepsilon^{j\frac{b}{\omega}\cos\omega t} \sum_{-\infty}^{+\infty} (-j)^n J_n(\tfrac{b}{\omega}) \frac{\varepsilon^{jn\omega t}}{a + jn\omega} \tag{47}$$

The latter expression can be modified if one applies the relation

$$\varepsilon^{j\frac{b}{\omega}\cos\omega t} = \sum_{-\infty}^{+\infty} j^m J_m(\tfrac{b}{\omega})\, \varepsilon^{jm\omega t}$$

There follows that (47) can be written equivalently

$$y(t) = \sum_m \sum_n j^{m-n} J_m(\tfrac{b}{\omega}) J_n(\tfrac{b}{\omega}) \frac{\varepsilon^{j(m+n)\omega t}}{a + jn\omega}$$

This expression can be ordered in the form of an exponential Fourier series of type

$$y(t) = \sum_{-\infty}^{+\infty} G_k\, \varepsilon^{jk\omega t} \tag{48}$$

where

$$G_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(\omega t)\, \varepsilon^{-jk\omega t}\, d\omega t$$

We have, for instance,

$$G_o = \sum_m \sum_n j^{m-n} J_m(\tfrac{b}{\omega}) J_n(\tfrac{b}{\omega}) \frac{\sin(m+n)\pi}{(m+n)\pi\,(a+jn\omega)}$$

Since in this expression

$$\frac{\sin(m+n)\pi}{(m+n)\pi} = \begin{cases} 0 \\ 1 \end{cases} \text{ if } m+n \begin{cases} \neq 0 \\ = 0 \end{cases}$$

there follows that $m = -n$ and

$$G_o = \sum_{-\infty}^{+\infty} J_n^2 (\tfrac{b}{\omega}) \frac{1}{a+jn\omega} \tag{49}$$

To derive this formula we have used the relations

$$j^{-2n} = (-1)^n, \quad J_n \cdot J_{-n} = (-1)^n J_n^{\,2}$$

Equation (49) can also be written

$$G_o = J_o^{\,2} \frac{1}{a} + \sum_1^{\infty} J_n^{\,2} \frac{2a}{a^2 + n^2\omega^2}$$

On the other hand, more generally we have

$$G_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(\omega t)\, \varepsilon^{-jk\omega t} d\omega t = \sum_m \sum_n j^{m-n} J_m(\tfrac{b}{\omega}) J_n(\tfrac{b}{\omega}) \frac{\sin(m+n-k)\pi}{(a+jn\omega)\pi(m+n-k)}$$

Since

$$\frac{\sin(m+n-k)\pi}{(m+n-k)\pi} = \begin{cases} 0 \\ 1 \end{cases} \text{if } m+n-k \begin{cases} \neq 0 \\ = 0 \end{cases}$$

there follows m+n = +k and

$$G_k = \sum_{-\infty}^{+\infty} j^{k-2n} J_{k-n}(\tfrac{b}{\omega}) J_n(\tfrac{b}{\omega}) \frac{1}{a+jn\omega}$$

In particular

$$G_1 = j \frac{J_o J_1}{a} + j \,{}_{1}^{\infty} J_n \frac{-J_{1-n}(a-j\omega)+J_{1+n}}{a^2 + \omega^2 n^2}$$

In the coefficients of the series (48), there appear Bessel functions of first kind, of argument $\varrho = b/\omega$ where $b/\omega \ll 1$. As a first approximation one can replace them with the limit value

$$\lim J_n(\varrho) = \frac{\varrho^n}{n!\, 2^n}$$

$$J_o(\varrho) \to 1, \quad J_n(\varrho) \to \frac{\varrho}{2}$$

We shall now derive the approximate solutions of equation (46) obtained by application of methods of iteration and of perturbation.

<u>Method of Iteration</u>. We write Eq. (46) as follows

$$\frac{dy(t)}{dt} + ay(t) = 1 - jb\, s(t)\, y(t) \tag{50}$$

Operating the change of variable bt = $\tau$, there follows

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = b\frac{d}{d\tau}$$

and Eq. (50) becomes

$$b\frac{d}{d\tau}y(\tau) + ay(\tau) = 1 - jb\, s(\tau)\, y(\tau)$$

i.e.,

$$\frac{d}{d\tau}y + \frac{a}{b}y = \frac{1}{b} - j\, s(\tau)\, y$$

By assumption $|s(t)| \leq 1$ and we assume that for our purposes the term $s(t).y$ can be considered small. Letting

$$y(t) = y_o(t) + y_1(t) + y_2(t) + \dots + y_n(t) + \dots.$$

where $y_n(t)$ is of order of $|s(t)|^n$, substituting into Eq. (50) and equating terms of the same order, we have the system of linear differential equations

$$\dot{y}_o + \frac{a}{b}y_o = \frac{1}{b} \tag{51}$$

$$\dot{y}_1 + \frac{a}{b}y_1 = -j\, s(\tau)\, y_o \tag{52}$$

$$\dot{y}_2 + \frac{a}{b}y_2 = -j\, s(\tau)\, y_1 \tag{53}$$

From Eq. (51) the steady state solution is

$$y_o(t) = \frac{1}{b}$$

There follows from (52)

$$\dot{y}_1 + \frac{a}{b}y_1 = -j\frac{s(\tau)}{b}$$

i.e.,

$$y_1(t) = -j \frac{\varepsilon^{-\frac{a}{b}\tau}}{a} \int \varepsilon^{\frac{a}{b}\tau} s(\tau)\, d\tau \tag{54}$$

Substituting (54) into Eq. (53), we have

$$\dot{y}_2 + \frac{a}{b} y_2 = - s(t) \frac{\varepsilon^{-\frac{a}{b}\tau}}{a} \int \varepsilon^{\frac{a\tau}{b}} s(\tau)\, d\tau$$

There follows

$$y_2(t) = - \frac{\varepsilon^{-\frac{a}{b}\tau}}{a} \int . s(\tau) \int \varepsilon^{+\frac{a\tau}{b}} s(\tau)\, d\tau\, d\tau$$

etc. For instance, if $s(t) = \sin \omega t = \sin \frac{\omega}{b}\tau$, we have

$$y_1(t) = -j \frac{\varepsilon^{-\frac{a}{b}\tau}}{b} \int \varepsilon^{\frac{a}{b}\tau} \sin \frac{\omega}{b}\tau d\tau = -j \frac{a \sin \omega t - \omega \cos \omega t}{\frac{a}{b}(a^2 + \omega^2)}$$

There follows to the first approximation

$$y(t) \cong y_o + y_1 = \frac{1}{a} - j \frac{b \sin \omega t}{a^2 + \omega^2} + j \frac{\omega b \cos \omega t}{b(b^2 + \omega^2)}$$

$$y_2(t) = \frac{-b^2}{2a(a^2 + \omega^2)} + \frac{a\, b \cos 2\omega t + 2\omega b \sin 2\omega t}{2a(a^2 + \omega^2)(a^2 + 4\omega^2)} +$$

$$+ \frac{\omega b( ab \sin 2\omega t - 2\omega b \cos 2\omega t)}{2a(a^2 + \omega^2)(a^2 + 4\omega^2)}$$

Therefore, the second iteration brings a contribution to the dc term and, in addition, terms of second harmonic.

Method of Perturbation. In order to apply the perturbation method in Eq. (46), we multiply the perturbation term $-jbs(t)\, y(t)$ by $\mu$, where $\mu$ is a parameter indicative of the order of magnitude of the perturbation term. At the end of our computations we shall replace it with one.

In this analysis no study is made of the convergence of the series obtained as a solution. Following the usual procedure, it is assumed that, even if the series obtained are not convergent, the first terms give a result close enough to the desired correct solution.

Now we expand y(t) into a power series in terms of $\mu$

$$y(t) = \sum_{0}^{\infty} y_n(t) \frac{\mu^n}{n!}$$

Replacing this series into Eq. (46), we have

$$\sum_{1}^{\infty} \frac{\mu^n}{n!} \dot{y}_n(\tau) + \frac{a}{b} \sum_{0}^{\infty} \frac{\mu^n}{n!} y_n(\tau) = \frac{1}{b} - j\mu s(t) \sum_{0}^{\infty} \frac{\mu^n y_n(t)}{n!} \tag{55}$$

Now letting $\mu = 0$, we have the equation of the zero approximation

$$\dot{y}_o(\tau) + \frac{a}{b} y_o(\tau) = \frac{1}{b} \tag{56}$$

Differentiating Eq. (55) once and then letting $\mu = 0$, we have the equation of the first approximation

$$\dot{y}_1(\tau) + \frac{a}{b} y_2(\tau) = - j\, s(\tau)\, y_o(\tau) \tag{57}$$

Similarly, differentiating twice Eq. (55) and then letting $\mu = 0$ we have the differential equation of the second approximation

$$\dot{y}_2 + \frac{a}{b} y_2 = - j\, 2\, s(\tau)\, \dot{y}_1(\tau) \tag{58}$$

It should be observed that Eqs. (56), (57), and (58) are similar to Eqs. (51), (52), and (53) of the iteration method. The same solutions apply in this case also.

<u>Other Methods of Iteration.</u> The given differential Eq. (46) may be written in the form

$$y(t) = \frac{1}{a+jbs(t)} - \frac{1}{a+jbs(t)}\, y'(t)$$

and solved by iteration assuming

$$y_n = \frac{1}{a+jbs(t)} - \frac{1}{a+jbs(t)}\, \dot{y}_{n-1}$$

As solution $y_o$ one assumes

$$y_o = \frac{1}{a+jbs(t)}$$

Then there follows

$$y_1(t) = \frac{1}{a+jbs(t)} + \frac{jbs'(t)}{[a+jbs(t)]^3}$$

and

$$y_2(t) = \frac{1}{a+jbs(t)} + \frac{jbs'(t)}{[a+jbs(t)]^3} - \frac{3b^2\ s'^2(t) + b[a+jbs(t)]s''(t)}{[a + jbs(t)]^5}$$

For instance if $s(t) = \sin \omega t$, we have

$$y_2(t) = \frac{1}{a+jb\ \sin \omega t} + \frac{jb\ \omega \cos \omega t}{(a+jb\ \sin \omega t)^3} +$$

$$- \frac{3\omega^2 b^2 \cos^2 \omega t - jb(a+jb\ \sin \omega t)\omega^2 \sin \omega t}{(a + jb\ \sin \omega t)^5}$$

Another procedure of approximate solution is obtained by writing the differential equation in integral form

$$y(t) = t - \int_0^t (a + jbs(t)\ \ y(t)\ dt$$

Successive solutions are obtained by iteration writing

$$y_n = t - \int_0^t [a + jbs(t)]\ y_{n-1}(t)\ dt$$

There follows

$$y_o = t$$

$$y_1 = t - \int_0^t [a + jbs(t)]\ t\ dt$$

$$y_2 = t - \int_0^t [a + jbs(t)] \left\{ t^2 - \int_0^t [a + jbs(t)]\ t\ dt \right\} dt$$

For example, when $s(t) = \sin \omega t$ one has

$$y_n(t) = t - \int_0^t (a + jb\ \sin \omega t) \left\{ t - at - j\frac{b}{\omega^2} (\sin \omega t - \omega t \cos \omega t) \right\} dt$$

$$= t - \frac{t^2}{2}(a-a^2) + j\frac{ab}{\omega^3}(1 - \cos\omega t) - j\frac{ab}{\omega^3}(\cos\omega t + \omega t \sin\omega t - 1)$$

$$- j\frac{b}{\omega^2}(\sin\omega t - \omega t\cos\omega t) + j\frac{ab}{\omega^2}(\sin\omega t - \omega t\cos\omega t) +$$

$$+ \frac{b^2}{2\omega^2}\left(t - \frac{\sin 2\omega t}{2}\right) - \frac{b^2}{4\omega^3}(\sin 2\omega t - 2\omega t\cos 2\omega t).$$

Another example is provided by the differential equation

$$\frac{dy}{dt} + a(y + by^3) = A\cos\omega t$$

which describes the behavior of a circuit consisting of a nonlinear resistance in series with a linear inductance. If $b \ll 1$, one can proceed by iteration considering y as the limit of a succession and letting

$$\frac{dy_n}{dt} + ay_n - A\cos\omega t = aby_{n-1}^3$$

If $y(o) = 0$, one has

$$y_o = \frac{C}{a}\varepsilon^{-at} + \frac{A}{a^2+\omega^2}\cos(\omega t - \varphi)$$

where $\tan\varphi = \frac{\omega}{a}$, $C = -\frac{a\,A}{a^2+\omega^2}\cos\varphi$

Similarly, $y_1$ is the solution of the differential equation

$$\frac{dy_1}{dt} + ay_1 - A\cos\omega t = ab\left\{\frac{C}{a}\varepsilon^{-at} + \frac{A}{a^2+\omega^2}\cos(\omega t - \varphi)\right\}^3$$

etc.

A different procedure for the solution of the given differential equation consists in transforming it first into the integral equation.

$$y = \frac{A}{\omega}\sin\omega t - a\int_o^t (y + by^3)\,dt$$

This equation can be solved by iteration letting

$$y_n = \frac{A}{\omega} \sin \omega t - a \int_0^t (y_{n-1} + b y_{n-1}^3)\, dt$$

one has, with the initial condition $y(o) = 0$:

$$y_o = \frac{A}{\omega} \sin \omega t$$

$$y_1 = \frac{A}{\omega} \sin \omega t - a \int_0^t \left[ \frac{-A}{\omega} \sin \omega t + b\left(\frac{A}{\omega}\right)^3 \sin^3 \omega t \right] dt =$$

$$= \frac{A}{\omega} \sin \omega t - \frac{aA}{\omega^2} (1 - \cos \omega t) -$$

$$- \frac{ab}{4} \left(\frac{A}{\omega}\right)^3 \left\{ \frac{3}{\omega} (1 - \cos \omega t) - \frac{a}{3\omega} (1 - \cos 3\, \omega t) \right\}$$

etc.

## VI. Difference Equations and Methods of Numerical Integration

### Difference Equations

Difference equations have particular importance for nonlinear analysis because they are used in numerical methods and in some analytical procedures of approximate integration; in addition, they are also used in describing the behavior of on-off automatic control systems. We shall limit ourselves here only to the treatment of linear difference equations.

While in differential calculus, which deals with quantities varying continuously in a certain range, one defines the differential operator $D = d/dt$ and the successive operators $D^2, \ldots D^n, \ldots$ in the calculus of finite differences, which deals with quantities varying discontinuously in a certain range, one defines the difference operator

$$\Delta y_n = y_{n+1} - y_n$$

and the successive difference operators

$$\Delta^2 y_n = \Delta^2 y_{n+1} - \Delta^2 y_n, \quad \ldots \quad \Delta^2 y_n = \Delta^r y_{n+1} - \Delta^r y_n$$

In terms of $\Delta$ one defines also the operator $E = \Delta + 1$, which satisfies the following operational relations

$$E\, y_n = y_{n+1}, \quad E^{-1}\, y_{n+1} = y_n, \ldots E^2\, y_n = E\, y_{n+1} = y_{n+2}, \quad \text{etc.}$$

Linear analytical relationships among variables of two corresponding sequences are expressed in form of difference equations of type

$$Q(\Delta)\, y_n = P(\Delta)\, v_n \tag{59}$$

where $P(\Delta)$, $Q(\Delta)$ are polynomials in $\Delta$ of type

$$Q(\Delta) = b_o\, \Delta^n + b_1\, \Delta^{n-1} + \ldots + b_n$$

$$P(\Delta) = a_o\, \Delta^m + a_1\, \Delta^{m-1} + \ldots + a_m$$

and $v_n$ represents a sequence of known values. The general solution of equation (59) can be expressed as sum of the solution of the homogeneous equation obtained from (59) by letting $P(\Delta) = 0$, and a particular solution of (59). The solution of a homogeneous difference equation of type

$$(b_o\, \Delta^r + b_1\, \Delta^{r-1} + \ldots + b_r)\, y_n = 0 \tag{60}$$

has the form

$$y_n = \sum_{i=1}^{r} A_i\, e^{\gamma_i n} \tag{61}$$

where the values of $\gamma_i$ are the roots of the transcendental equation obtained introducing (61) into (60). One can also express Eq. (60) by means of the operator E and obtain a relationship of type

$$b_r\, y_{n+r} + b_{r-1}\, y_{n+r-1} + \ldots + b_o\, y_n = 0 \tag{62}$$

The solution of the latter equation has the form

$$y_n = \sum_{i=1}^{r} A_i\, \rho_i^{\,n}$$

where $\rho_i$ are the roots of the auxiliary equation

$$b_r\, \rho^r + \ldots + b_o = 0$$

Because Eq. (59) has been assumed to be linear, the principle of superposition applies. Accordingly, it is possible to express its solution by means of an integral expression similar to the integral of Duhamel. For this purpose, one defines the unit step sequence $u_n$ as

$$u_n = 0 \quad \text{for } n = 0$$
$$u_n = 1 \quad \text{for } n > 0 \tag{63}$$

and the unit impulse sequence $u'_n = \Delta u_n$ as

$$u'_n = \Delta u_n = 0 \quad \text{for } n \neq 0$$
$$u'_n = \quad 1 \quad n = 0 \tag{64}$$

Then by application of linear superposition, one can express any bounded function $y_n$ in terms of unit step or unit impulse sequence as follows:

$$y_n = y_o u_n + \sum_{r=1}^{n} \Delta y_r u_{n-r} \tag{65}$$

or

$$y_n = \sum_{o}^{n} y_r u'_{n-r} \tag{66}$$

Since

$$u_{n-r} = E^{-r} u_n, \quad u'_{n-r} = E^{-r} u'_n$$

equations (7) and (8) can be written respectively

$$y_n = \left[ y_o + \sum_{r=1}^{n} \Delta y_r E^{-r} \right] u_n =$$

$$= \left[ y_o + \Delta y_1 E^{-1} + \Delta y_2 E^{-2} + \ldots \right] u_n = y(E).u_r \tag{65'}$$

$$y_n = \sum_{r=0}^{n} y_r E^{-r} u'_n = \left[ y_o + y_1 E^{-1} + \ldots . \right] u'_n =$$

$$= y^* (E) u'_n \tag{66'}$$

The coefficients of $Y(E)$ and of $Y^*(E)$ can be read directly from a graph of the sequence $y_n$ versus time.

If we now indicate with $A_n$ and $G_n$, respectively, the solutions of Eq. (59) for $v_n = u_n$ and $v_n = u'_n$, the general solution of the same equation, can be written as follows:

$$y_n = \left[ v_o + \sum_{r=1}^{n} \Delta v_r E^{-r} \right] A_n \tag{67}$$

$$y_n = \sum_{r=0}^{n} v_r \, E^{-r} \, G_n \tag{68}$$

The functions $A_n$ and $G_n$ are particular integrals of Eq. (59) which can be written symbolically

$$A_n = \frac{P(\Delta)}{Q(\Delta)} \, u_n$$

$$G_n = \frac{P(\Delta)}{Q(\Delta)} \, u'_n$$

By long division, letting $\Delta = E - 1$, and assuming that $m < n$, one has:

$$\frac{P(\Delta)}{Q(\Delta)} = \frac{a_o(E-1)^m + a_1(E-1)^{m-1} + \ldots + a_m}{b_o(E-1)^n + b_1(E-1)^{n-1} + \ldots + b_n} = C_o + C_1 E^{-1} + C_2 E^{-2} + \ldots$$

Therefore, $A_n$ and $G_n$ can be evaluated and expressed in the forms (65') and (66'), respectively. From these expressions one obtains the general solution $y_n$ corresponding to the arbitrary sequency $v_n$. However, the latter can also be obtained more directly by letting $v_n = V(E).U_n$ or $v_n = V^*(E).U'_n$ in Eq. (59). One has respectively

$$y_n = \frac{P(E-1).V(E)}{Q(E-1)} \, u_n$$

$$y_n = \frac{P(E-1).V^*(E)}{Q(E-1)} \, u'_n$$

## Numerical Integration

One of the simplest methods of numerical integration is based on direct use of Taylor's series. Given

$$y' = f(x,y) \tag{69}$$

with $y(x_o) = y_o$ one first evaluates from (69) $y'(x_o)$. Now differentiating Eq. (69) successively one evaluates $y''$, $y'''$, etc., and correspondingly, $y''(x_o)$, $y'''(x_o)$, etc. There follows, choosing h small enough,

$$y(x_1) = y(x_o+h) = y(x_o)+hy'(x_o)+ \frac{h^2}{2} y''(x_o)+ \ldots$$

$$y'(x_1) = y'(x_o+h) = y'(x_o)+hy''(x_o)+ \frac{h^2}{2} y'''(x_o)+ \ldots$$

Having obtained $y(x_1)$ and $y'(x_1)$ one evaluates from Eq. (69) similarly $y''(x_1)$, $y'''(x_1)$, etc., and obtains $y(x_2) = y(x_1 + h)$. The process is repeated until the required solution is obtained. A check on the accuracy of the computations is made by adding separately the terms of odd power and those of even power of each series. Then, if the series corresponds to $x = x_n$, the sum of these two results gives $y(x_{n+1})$ and their difference gives $y(x_{n-1})$. The latter value should coincide with the value previously calculated, and provides an indication of the error and a correction term.

In particular the method of Euler* as applied to the differential Eq. (69) consists in using a Taylor expansion limited to the linear term, i.e.,

$$y(x_n) = y(x_{n-1}) + \left(\frac{dy}{dx}\right)_{n-1} \cdot h$$

An improved value of $\left(\frac{dy}{dx}\right)_{n-1}$ is found at each step by multiplying h by the average of the values of (dy/dx) at the ends of the interval $x_{n-1}$, $x_n$, i.e.,

$$\frac{1}{2}\left[\left(\frac{dy}{dx}\right)_{n-1} + \left(\frac{dy}{dx}\right)_n\right]$$

A simplified procedure of numerical integration is obtained replacing dy/dx in Eq. (69) by an approximating polynomial and then integrating this over any desired interval. For this purpose, one uses the formula of Newton

$$y' = y'_n + \Delta y'_n u + \frac{\Delta^2 y'_n}{2}(u^2+u) + \frac{\Delta^3 y'_n}{6}(u^3+3u^2+2u) + \frac{\Delta^4 y'_n}{24}(u^4+6u^3+11u^2+6u). \tag{70}$$

where $u = (x-x_n)/h$, and $\Delta^n$ are difference operators of order n. Integrating Eq. (70) over the intervals $x_n \div x_{n+1}$, $x_{n-1} \div x_n$, etc., one obtains various formulas for integrating ahead or for checking and improving previously calculated values. For example, one has

$$\int_{x_n}^{x_{n+1}} y'dx = h\left[y'_n + \frac{1}{2}\Delta' y'_n + \frac{5}{12}\Delta^2 y'_n + \frac{3}{8}\Delta^3 y'_n + \frac{251}{720}\Delta^4 y'_n\right]$$

$$\int_{x_{n-1}}^{x_n} y'dx = h\left[y'_n - \frac{1}{2}\Delta' y'_n - \frac{1}{12}\Delta^2 y'_n - \frac{1}{24}\Delta^3 y'_n - \frac{19}{720}\Delta^4 y'_n\right] \tag{71}$$

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* J. B. Scarborough, Numerical Mathematical Analysis, The John Hopkins Press, 1950, p. 234.

In order to use these formulas one needs to start the solution; since the starting values must be very accurate, this part is usually very laborious. The most common method used to start the solution is based on the use of Taylor's series, as already described.

The approximate numerical solution of a differential equation can also be obtained by replacing directly the equation with its equivalent difference equation. To do this, one observes that by application of Taylor's series for h small enough

$$y(x+h) = \left[1+hD+\frac{h^2}{2!}D^2+\dots\right]y(x) = \varepsilon^{hD}h(x)$$

and similarly in finite differences

$$y(x_n+h) = h(x_{n+1}) = Ey(x_n) = \varepsilon^{hD}y(x_n)$$

There follows

$$E = \varepsilon^{hD} = 1+hD+\frac{h^2}{2!}D^2+\dots$$

and

$$D = \frac{1}{h}\ln E = \frac{1}{h}\ln(1+\Delta) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right] \tag{72}$$

An approximate representation of Eq. (72) in terms of $E^{-n}$ is

$$D \simeq \frac{1}{h}\ \frac{1-E^{-n}}{P_n(E^{-1})} \tag{73}$$

where $P_n(E^{-1})$ is a polynomial of degree n, whose coefficients might be obtained by substitution. For instance, it is found

$$n = 1 \qquad D \simeq \frac{2}{h}\ \frac{1-E^{-1}}{1+E^{-1}}$$

$$n = 2 \qquad D \simeq \frac{3}{h}\ \frac{1-E^{-2}}{1+4E^{-1}+E^{-2}}$$

$$n = 3 \qquad D \simeq \frac{8}{3h}\ \frac{1-E^{-3}}{1+3E^{-1}+3E^{-2}+E^{-3}}$$

The errors of these expressions depend upon the third, fourth and fourth difference, respectively. Another less accurate representation of D which is often used is from (14)

$$D \simeq \frac{\Delta}{h} \tag{74}$$

Given a differential equation

$$Q(D)y = P(D)v(t)$$

one writes

$$y = \frac{P(D)}{Q(D)} v(t) = F(D).v(t) \tag{75}$$

and replaces the powers of D with the corresponding approximate expressions in $E^{-n}$. If the highest order of D is m, the approximate expression used for D should be accurate to the m-th difference.**

For example, consider the differential equation

$$(D + 1)\, y = u(t)$$

One has

$$y = \frac{1}{1+D} u(t) = \frac{1}{1 + \frac{2}{h}\,\frac{1-E^{-1}}{1+E^{-1}}} u_n$$

$$y_n = h\, \frac{1 + E^{-1}}{2+h-(2-h)E^{-1}}\, u_n$$

This expression can be evaluated by means of continuous division.

The application of the previous methods to differential equations of higher order presents no difficulty since these can be readily transformed into a system of equations of first order.

The solution of difference equation can also be obtained by means of the so-called relaxation method.* The difference equation (given or obtained by transformation from a differential equation) is written preferably in the form (62), i.e.,

$$b_r\, y_{n+r} + b_{r-1}\, y_{n+r-1} + \ldots + b_o\, y_n = 0 \tag{62}$$

This equation connects sets of r consecutive points. We might assume, for example, that y represents values of potential in a region with given boundaries. To begin with, one selects a rather wide square net covering the given region, and whose intersection points are the values $x_n$. One fills the entire area with guessed-at values of y, and as a result the application of the Eq. (62) now will provide a residual different than zero, i.e.,

$$b_r\, y_{n+r} + b_{r-1}\, y_{n+r-1} + \ldots + b_o\, y_n = R(n) \tag{76}$$

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* E. Weber, Electromagnetic Fields, Theory and Applications, J. Wiley, 1950, p. 260.

** Milne-Thomson - Calculus of finite differences, London, 1923. Brown, B.M. - Math. Gazette 30 (1947), 145.

For the exact solution R(n) should be zero, and its actual value depends upon the initial value $y_n$ assumed. Any correction at n will affect the residuals at all its neighbor points as well. It requires therefore some little experience to estimate the corrections needed, and it is generally preferable to note next to the assumed values $y_n$ the residuals in brackets. One will use the distribution of the residuals for the second estimate. It is important to note that the procedure is definitely a convergent one even if one starts from a rather crude first guess. Good results are recorded by Strutt* for electron tube problems. The method is illustrated in Cosslett** and Zworkyn, et al*** for electron optical problems and in Southwell**** for the magnetic flux distribution in a generator; many applications have been made to elastic and heat problems.

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* M. J. O. Strutt, Ann. d. Physik 87, p. 153 (1928). M. J. O. Strutt, Moderne Mehrgitter-electronenröhren, Vol. 2, J. Springer, Berlin, 1938.

** V. E. Cosslett, Introduction to Electron Optics, Oxford University Press, England, 1946.

*** V. K. Zworkyn, et al, Electron Optics and the Electron Microscope, J.Wiley and Sons, 1945.

**** R. V. Southwell, Relaxation Methods in Theoretical Physics, Oxford Univ. Press, 1946.